

What is in a pebble shape?

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We propose to characterize the shapes of flat pebbles in terms of the statistical distribution of curvatures measured along the pebble contour. This is demonstrated for the erosion of clay pebbles in a controlled laboratory apparatus. Photographs at various stages of erosion are analyzed, and compared with two models. We find that the curvature distribution complements the usual measurement of aspect ratio, and connects naturally to erosion processes that are typically faster at protruding regions of high curvature.

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In soft matter physics it is well understood why certain objects have naturally rounded shapes. For example, liquid droplets are round due to surface tension [1] and bilayer vesicles are round due to bending rigidity [2]. In both cases the equilibrium shapes are determined by free-energy minimization, and sinusoidal perturbations decay exponentially with time according to a power of the wavevector. But how do pebbles in a river or on a beach come to have their familiar, smooth, rounded shapes [3]? Are there similarly simple physical principles that govern the evolution of freshly-fractured polyhedral rocks into smooth rounded pebbles? Aristotle proposed that erosion is more rapid at regions farther from the center, where greater impulses can be more readily delivered [3]. However this idea has not been quantified or tested, and we are aware of no other models for the origin of pebble shapes.

A wealth of knowledge already exists on naturally-occurring pebbles [4], thanks to decades of field and laboratory study [5, 6, 7, 8, 9, 10]. There is a rich terminology for describing sizes (grain, pebble, cobble, boulder) and shapes (angular, rounded, elongated, platy). The simplest quantitative shape description is to form dimensionless indices from the lengths of long vs intermediate vs short axes. There is still debate as to how these axes are best defined, as well as to which dimensionless ratios are most useful [11, 12, 13, 14, 15]. Another quantitative method used more for grains than for flat pebbles involves Fourier transform of the contour [4, 16, 17, 18]. There the relative amplitudes of different harmonics give an indication of shape in terms of roughness at different length scales. However a Fourier description is not particularly natural because peaks erode faster than valleys, so that sinusoidal perturbations cannot erode without changing shape. To make progress in understanding the origin of pebble shapes, one needs a quantitative shape description in terms of microscopic variables directly relevant to the erosion process.

In this paper we propose to quantify the shape of flat pebbles in terms of *curvature*, measured at each point

along the entire two-dimensional contour. The curvature K equals the reciprocal radius of a circle that locally matches the contour, and can be deduced from the coordinates of the pebble boundary [19]. This could be useful, since one can imagine faster erosion at protruding regions of higher positive curvature, and perhaps no erosion for scalloped regions of negative curvature. To illustrate the use of curvature for shape description, we perform a laboratory experiment in which clay pebbles are eroded from controlled initial shapes. We find that the curvature distribution along the pebble contour becomes stationary, indicating a final fixed-point shape. Surprisingly, this stationary shape is not a perfect circle. We then introduce two models of pebble erosion, and test them using the curvature distribution as well as the aspect ratio. We find that the curvature distribution is a more incisive tool for discriminating between models. These results point the way for future studies, both of naturally-occurring pebbles and of lab-eroded pebbles of more direct geophysical interest.

For our laboratory experiments, the pebbles were prepared from ordinary white clay (“chamotte”), molded into different polygonal shapes with uniform 0.5 cm thickness. Individual pebbles were eroded one at a time in a square pan, 30×30 cm² with 7 cm walls, spun at 1 Hz around the central axis perpendicular to the bottom surface and oriented at 45° away from vertical. The pebble is first dragged upwards, at rest on the pan; near the top it begins to slide, and it accelerates until striking the wall; then it rolls along the side, comes to rest, and starts a new cycle. Erosion is mainly caused by collision with the walls. At five minute intervals, corresponding to roughly 300 collision events, the clay pebble is removed and photographed. As an example, photographs of one initially-square pebble with sides of 5 cm are shown in Fig. 1; digitized contours are shown in the inset of Fig. 2a. Evidently the erosion is fastest at the corners, which protrude and have high positive curvature. Once the corners have been removed, the pebble reaches a nearly round shape that progressively shrinks.

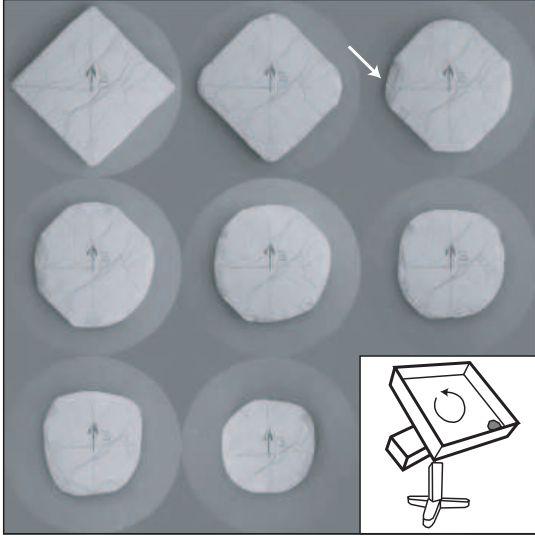


FIG. 1: Shape evolution of a $5 \times 5 \times 0.5$ cm³ square pebble eroded in a rotating basin, depicted on the lower right, with roughly 300 cutting events between each photograph. The time sequence is left to right, and top to bottom. The erosion is due to small random cutting events, a particularly large example of which is highlighted by arrow in the third photograph.

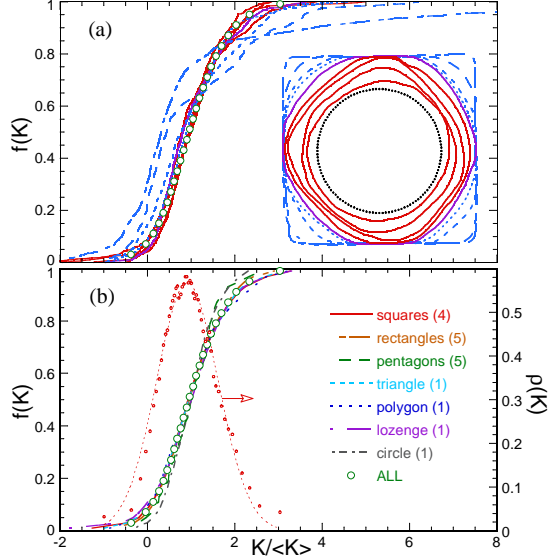


FIG. 2: [Color online] (a) Digitized contours (inset) and cumulative curvature distribution for the initially-square clay pebble of Fig. 1. Note that the red curves are indistinguishable, indicating a stationary final shape. The final shape is not circular, as demonstrated both by the black-dotted circle in the inset and by the gradual rise of $f(K)$ from zero to one. (b) Left axis: Cumulative curvature distribution for the final shapes attained by clay pebbles of varied initial shape, as labeled; the number of runs for each shape is given in parentheses. Right axis: average curvature distribution for all final shapes, $\rho = df/dK$, and best fit to a Gaussian.

The first step in analysis is to deduce the value of the curvature at each digitized point along the contour by linear-least-squares fits of radius vs angle to a third-order polynomial. Some care is needed in choice of fitting window: if it is too small then the matrix inversion of the fit is singular [20], if it is too large then the fit deviates systematically from the contour points; results for all acceptable windows are averaged together. As a check, the perimeter P is computed from straight-line segments between adjacent points and is confirmed to equal 2π divided by the average curvature $\langle K \rangle$ [19].

As a shape-descriptor, we next construct the curvature distribution such that $\rho(K)dK$ is the probability of finding a value of the curvature between K and $K + dK$. For reliability, it is preferable to work with the cumulative curvature distribution, $f(K) = \int_0^K \rho(K')dK'$, which may be computed as the fraction of the perimeter with curvature less than K [20]. For a perfect circle, $\rho(K)$ is a delta-function and $f(K)$ is a step-function. The evolution of $f(K)$ for one square pebble is shown in the main plot of Fig. 2a, corresponding to the contours of the inset. Note that the results are plotted vs $K/\langle K \rangle$ in order to remove a trivial scale factor related to the ever-shrinking perimeter. Evidently, the curvature distribution starts broad but narrows down as erosion proceeds. By about the fifth (purple) or sixth (largest red) contour, the form of $f(K)$ vs $K/\langle K \rangle$ reaches a steady state that fluctuates around a well-defined average. If the initial shape is circular, then by contrast the form of $f(K)$ starts narrow but broadens as erosion proceeds.

To test the possible influence of initial conditions on the final shape, we repeat the erosion experiment of Fig. 2a for clay pebbles molded into several different shapes. In all cases the form of $f(K)$ vs $K/\langle K \rangle$ eventually becomes time independent, with results given for different initial shapes on the left axis of Fig. 2b. To within fluctuations comparable with those seen for a single stationary pebble as it shrinks further, these are all identical. Thus the final shape is truly *stationary*, independent of both size and the initial conditions, and may be quantified by averaging together the final cumulative curvature distributions for all pebbles; the results are shown by open circles in Fig. 2b. Note that its form is not set by uncertainty in curvature measurement, since the circular pebble started with a narrower distribution.

The stationary curvature distribution, obtained by numerical differentiation of the average $f(K)$, is shown on the right axis of Fig. 2b. Clearly it is not a delta-function, and hence the shape is not a perfect circle. The form of $\rho(K)$ can be reasonably fit to a Gaussian, as shown, with a standard deviation of $\sigma/\langle K \rangle = 0.7$. However, the actual distribution is skewed in the tails and has a standard deviation of $\sigma/\langle K \rangle = 0.8$. Though our laboratory erosion experiment may be artificial from a geophysical perspective, it is interesting that a stationary shape exists and that it is not trivially circular. Clearly, $f(K)$ was a useful

tool for this demonstration.

In the remainder of the paper, we use the cumulative curvature distribution to evaluate the validity of two simple models of the laboratory erosion experiment. We begin with a “*polishing model*” in which erosion is directly coupled to curvature. Namely, the normal velocity at each boundary point is taken in proportion to curvature, for regions of positive curvature, and is zero otherwise. (If all points are moved normally in proportion to curvature, irrespective of sign, then it has been proven that any initial contour will shrink to a point and approach a circle in this limit [21].) The evolution predicted by the polishing model for a square pebble is shown in Fig. 3. In the top plot, Fig. 3a, the contours and cumulative curvature distributions are plotted at the same sequence of perimeters, and with the same line-color codes, as the actual data in Fig. 2a. As expected, this model is highly efficient at polishing a rough pebble into a smooth shape. It becomes essentially circular, to the eye, soon after the entire original boundary has been eroded. The cumulative curvature distribution, $f(K)$, appears to approach a step function, contrary to our laboratory experiments shown in Fig. 2. Thus the polishing model may be appropriate for other natural or laboratory erosion processes, but not for ours.

To track evolution toward a final shape more clearly, and to compare new vs old shape description methods, the aspect ratio of the pebbles and the width of the curvature distributions are shown in Figs. 3b-c. These are plotted vs perimeter divided by the initial perimeter, as a surrogate for the duration of erosion (except that time evolution is from right to left). Since major and minor orthogonal axes are equal for a square shape, we work with the “caliper” aspect ratio, C/A , of largest to smallest separations for parallel planes that confine the pebble; as the name suggests, this quantity may be readily measured in the field using calipers. For a dimensionless measure of the width of the curvature distribution, we take the standard deviation, σ , divided by the average curvature, $\langle K \rangle$. The results for all pebbles are plotted in Figs. 3b-c, with four initially-square pebbles highlighted by larger symbols. Stationarity is reached when the square pebbles are eroded to about 3/4 of their initial perimeter, after which both C/A and $\sigma/\langle K \rangle$ data fluctuate significantly about an average value. These fluctuations represent real shape changes, not measurement uncertainty, and are larger for C/A than for $\sigma/\langle K \rangle$. Predictions by the polishing model for the evolution of the four square pebbles are also included for completeness. While the initial agreement is satisfactory, the polishing predictions for C/A and $\sigma/\langle K \rangle$ quickly fall below the data and, furthermore, do not exhibit fluctuations.

To more successfully explain the laboratory erosion, we introduce a minimal one-parameter “*cutting model*”. We imagine the effect of collision between pebble and wall is to create a straight fracture near the boundary

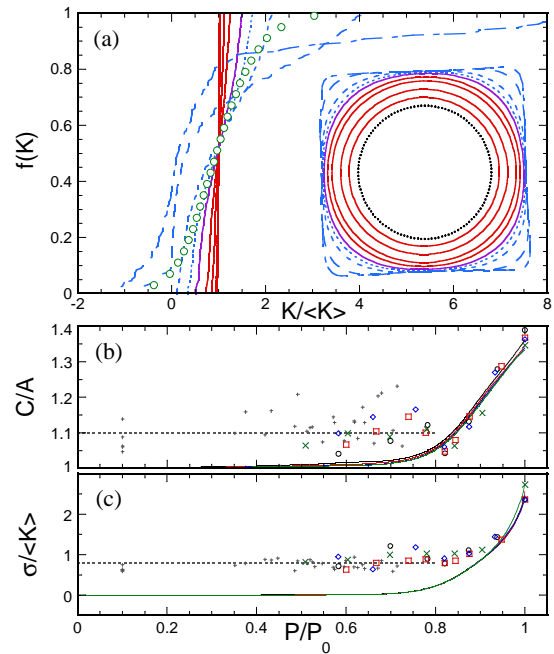


FIG. 3: [Color online] (a) Contours (inset) and cumulative curvature distribution predicted by the “*polishing model*” for a square pebble. The initial contour points, the sequence of perimeters displayed, and the line-color codes, are the same as in Fig. 2a. The open green circles represent the stationary $f(K)$ averaged over all laboratory pebbles. (b) The caliper aspect ratio and (c) the standard deviation of the curvature distribution vs scaled perimeter; note that time evolution is from right to left. Symbol and line codes for (a) and (b) are as follows: Data for four different square pebbles are shown by large colored symbols; data for pebbles of other initial shape are shown as small gray pluses once stationarity has been reached; for the five pentagons P_0 was not measured, so data are shown arbitrarily at $P/P_0 = 0.1$; aspect ratio and standard deviation for the final stationary shape of the laboratory experiments are shown by dashed lines. Predictions of the polishing model, applied to the four square pebbles, are shown by solid curves.

whose size, on average, is set by the collision impulse; e.g. note the unusually large cut in the third photograph in Fig. 1. At each erosion step of the model, we thus pick a contour point at random and we choose a cut length from an exponential distribution whose average is some fraction α of the square-root of the pebble area. One can imagine other choices, but an exponential is simple and the impulse is set by the area rather than the perimeter.

To match the cutting model with our data, we take $\alpha = 0.042$ in order that the final, stationary shape have the same curvature distribution width as the laboratory pebble data. The resulting evolution of the contours and curvature distributions predicted for one square pebble are shown in Fig. 4a. As observed in the lab, the corners erode faster than the flat sections and the final shape is nearly, but not perfectly, circular. The cutting model predictions are compared directly with the C/A

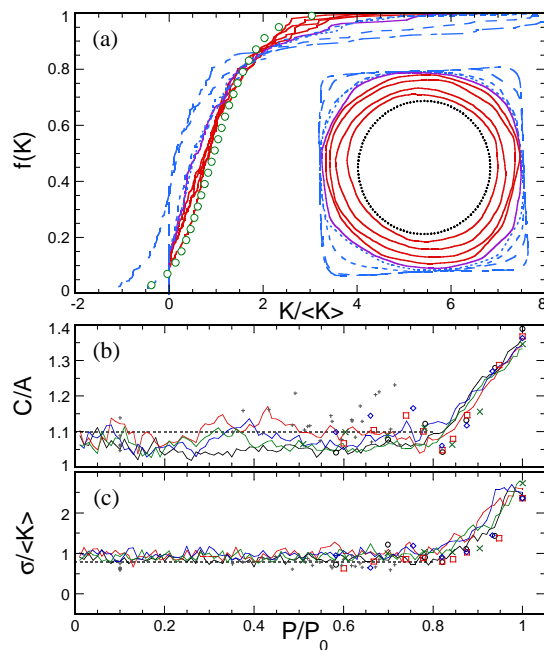


FIG. 4: [Color online] Predictions for evolution based on the “cutting model”, and comparison with data. Displayed quantities are the same as for the “polishing model” in Fig. 3.

and $\sigma/\langle K \rangle$ data in Figs. 4b-c. While the final average value of $\sigma/\langle K \rangle$ is correct by design, our model also captures several other features of the data: the final average value of C/A , the magnitude of shape fluctuations in the stationary shape, and the rate of evolution from initial to final shapes. In spite of this success, the full form of $f(K)$ in Fig. 4a is not completely correct in fine detail. Particularly, the model does not generate regions of negative curvature. No doubt this could be remedied by taking cuts from an appropriate distribution of shapes, not just straight. Such refinements could be tested by comparison with $f(K)$ data.

In conclusion, the distribution of curvature along the contour of a pebble is more incisive than the aspect ratio as a tool for both describing pebble shapes and for testing models. The characterization of such eroded forms is of obvious importance in the subfield of geology known as sedimentology [4], where pebbles are considered as witness to the geological conditions under which they were formed. It would therefore be of great practical consequence if there existed a method to decipher the message imprinted in a pebble shape, a method that would

distinguish beach vs glacier vs river erosion, that would tell how far has a pebble traveled down a stream and perhaps even for how long it has been subjected to erosion forces. Moreover, the ability to quantify and explain shapes resulting from abrasion, erosion or other wearing mechanisms would also be of significance in other fields, wherever nature or man transforms a solid object by irreversibly removing fractions of the mass in a sequence of elementary events. The curvature distribution could prove a helpful tool to study all such processes.

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